Lecture 23

Monday, December 2, 2019 9:43 AM

Recall. • $\gamma \sim 0$, $\gamma \approx 0$ and Cauchy's Thm I–II

Prop 1. Let $\gamma : [0,1] \to G \subseteq \mathbb{C}$ be closed, p-w smooth. Then, $\gamma \sim 0 \Rightarrow \gamma \approx 0$

Rem. Converse not true.

• Do FEP-homotopy + Indep. of Path Thm from Lecture 22 notes.

Def. An open set $G \subseteq \mathbb{C}$ is *simply connected* if G is connected and every closed γ is homotopic to 0 in G.

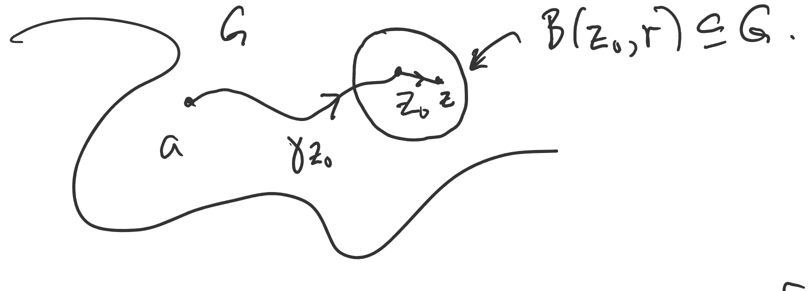
Cauchy's Thm IV. Let $G \subseteq \mathbb{C}$ be simply connected, and $\gamma : [0,1] \to G$ closed, p-w smooth. If f anal. in G, then

$$\int_\gamma f\,dz = 0.$$

Cor 1. Let G be simply connected.

1. If f is anal. in G, then $\exists F$ anal. s.t. $F' = f$.
2. If f is anal. and $f \neq 0$ in G, then there is anal. well-defined branch of $\log f$. I.e. $\exists g$ anal s.t. $f = e^g$.

Pf. ① Fix $a \in G$ and define for every $z \in G$, $F(z) = \int_{\gamma_z} f(\zeta)\,d\zeta$, where γ_z is a p-w smooth curve (even polygonal path) from a to z. This function is well-defined by I. of P. Thm. We claim that F is $\mathbb{C}$-diff and $F' = f$, which would complete the proof. Pick $z_0 \in G$



For $z \in B(z_0, r)$ we may take $\gamma_z = \gamma_{z_0} + [z_0, z] \Rightarrow$

For $z \in B(z_0, r)$ we may take $\gamma_z = \gamma_{z_0} + [z_0, z]$.

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0} \int_{[z_0,z]} \left(f(\zeta)-f(z_0)\right) d\zeta .$$

For any $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|f(\zeta)-f(z_0)| < \varepsilon$ if $|\zeta - z_0| < \delta$.

In particular if $|z-z_0| < \delta \Rightarrow \max\limits_{\zeta \in [z, z_0]} |f(\zeta) - f(z_0)| < \varepsilon \Rightarrow$

$$\left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right| < \varepsilon \Rightarrow F \text{ } \mathbb{C}\text{-diff at } z_0 \text{ w/ } F'(z_0) = f(z_0). \quad \blacksquare$$

(2) Let $H(z)$ be primitive of $\frac{f'(z)}{f(z)}$ in G. We claim $\underbrace{\exists C \neq 0 \text{ s.t. } G(z) = H(z) + C \text{ s.t. } e^{G(z)} = f(z)}$.

Consider $h(z) = e^{H(z)} \Rightarrow h' = H' e^{H} = \frac{f'}{f} \cdot h$ or $h' f = f' h$. Compute

$$\left(\frac{f}{h}\right)' = \frac{f'h - fh'}{h^2} = 0 \Rightarrow \frac{f}{h} = C, \text{ constant} \neq 0.$$

$$\Rightarrow f(z) = C e^{H(z)} = e^{H(z)+C} = e^{G(z)} \text{ as claimed. } \blacksquare$$

<u>Open Mapping Thm and counting zeros</u>

<u>Thm1.</u> Let f be anal. in $G \subseteq \mathbb{C}$, w/ zeros $a_1, \ldots, a_n$ (repeated w/ multi's). Then, if γ is closed, p-w smooth, $\gamma \approx 0$ in G, and no $a_k \in \{\gamma\}$, $\Rightarrow$ $\sum\limits_{k=1}^{n} n(\gamma, a_k) = \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta$

<u>Pf.</u> Recall that we may factor $f(z) = (z-a_1)\ldots(z-a_n) g(z)$ where g is anal. in G and $g \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z-a_k} + \frac{g'(z)}{g(z)} .$$

$$\frac{f'(z)}{f(z)} = \sum_{k=1} \frac{1}{z-a_k} + \frac{g'(z)}{g(z)}.$$

Since $\frac{g'}{g}$ analytic in G, we get by CT-I,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \sum_{k=1}^{n} \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - a_k}}_{n(\gamma, a_k)} + \cancel{\frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)\,d\zeta}{g(\zeta)}}^{\,0}. \quad \blacksquare$$

Cor 2. Let f be anal. in G, γ closed, p-w smooth, w/ $\gamma \approx 0$ in G. Let $a_1, \dots, a_n$ be solutions of $f(z) = \alpha$ and assume no $a_k \in \{\gamma\}$.

Then,
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta) - \alpha}\,d\zeta = \sum_{k=1}^{n} n(\gamma, a_k).$$

Counts # of roots, w/ multi + index of $f(z) = \alpha$ "inside" γ

Pf. Consider $g(z) = f(z) - \alpha$ and apply Thm 1. $\blacksquare$

Thm 2. Let f be anal. in G, $f(a) = \alpha$ w/ multi m $\left(f'(a) = \dots = f^{(m-1)}(a) = 0,\ f^{(m)}(a) \neq 0\right)$. Then, $\exists\, \varepsilon, \delta > 0$ s.t. $f(z) = \beta$ has exactly m simple roots in $B(a, \delta)$ for each $\beta \in B(\alpha, \varepsilon)$.

Pf. $\exists\, \delta > 0$ s.t. $f(z) \neq \alpha$, $f'(z) \neq 0$ in $\overline{B(a,\delta)} \setminus \{a\}$. Consider the closed, p-w smooth curve $\sigma = f \circ \gamma$, where $\gamma(s) = a + \delta e^{2\pi i t}$.

By assumption, $\alpha \notin \{\sigma\}$. Now,

$$n(\sigma, \alpha) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w - \alpha} = \left\{ \begin{array}{l} w = f(z),\ z \in \{\gamma\} \\ dw = f'(z)\,dz \end{array} \right\} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z) - \alpha} = \{\text{Thm 1}\}$$

$$= \sum_{k=1}^{n} n(\gamma, \underset{\substack{\uparrow \\ f(a_k) = \alpha}}{a_k}) = \left\{ \begin{array}{l} n(\gamma, a_k) = 1 \text{ if } a_k \in B(a, \delta) \\ \text{and } = 0 \text{ otherwise} \end{array} \right\} = m$$

By previous Thm, $n(\sigma,\beta)=m$, $\forall \beta$ in same component of $\mathbb{C}\setminus\{\sigma\}$ as α. Choose $\varepsilon>0$ s.t. $B(\alpha,\varepsilon)\cap\{\sigma\}=\emptyset$, then $n(\sigma,\beta)=m$, $\forall \beta\in B(\alpha,\varepsilon)$. But, $n(\sigma,\beta)=\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)-\beta} =$

$$\sum_{k=1}^{n} n(\gamma,b_k) = \# \text{ of solutions w/ multi to } f(z)=\beta \text{ in } B(a,\delta)$$

$f(b_k)=\beta$

Since $f'\neq 0$ in $B(a,\delta)\setminus\{a\}$, all roots are simple (multi $=1$).

$\boxtimes$